BULL. AUSTRAL. MATH. SOC. 05C50, 15A18
VOL. 69 (2004) [133–136]

CHARACTERISTIC POLYNOMIALS OF GRAPH COVERINGS

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In this note, a formula for the characteristic polynomial of any (regular or irregular) graph covering is described.

Let G be a finite simple graph with vertex set $V(G) = \{v_1, v_2 \ldots, v_m\}$. The *adjacency matrix* $A(G) = (a_{ij})$ is the $m \times m$ matrix with $a_{ij} = 1$ if v_i and v_j are adjacent and $a_{ij} = 0$ otherwise. The *characteristic polynomial* of G, denoted by $\Phi(G; \lambda)$, is the characteristic polynomial $\det(\lambda I - A(G))$ of $A(G)$.

A *covering projection* (or simply *covering*) from a graph $\widetilde{G}$ to another G is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p\,|_{N(\widetilde{v})} : N(\widetilde{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\widetilde{v} \in p^{-1}(v)$, where $N(v)$, the neighbourhood of v, is the set of vertices adjacent to v. Sometimes, a graph $\widetilde{G}$ is also called a covering of G with the projection $p : \widetilde{G} \to G$, and it is n-fold if p is n-to-one.

Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse directed edge to a directed edge $e = uv$. A directed edge is also called an *arc* and the set of arcs of the graph G is denoted by $D(G)$. Let S_n be the symmetric group on $\Omega = \{1, 2, \ldots, n\}$. A *voltage assignment* ϕ of G is a function $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *derived graph* G^ϕ from a voltage assignment ϕ is defined as $V(G^\phi) = V(G) \times \Omega$, and (u, i) and (v, j) are adjacent if $uv \in D(G)$ and $j = i^{\phi(uv)}$. The first coordinate projection $p_\phi : G^\phi \to G$ is an n-fold covering. Let $C^1(G; n)$ denote the set of all voltage assignments $\phi : D(G) \to S_n$ of G. Gross and Tucker [**2**] showed that every n-fold covering $\widetilde{G}$ of a graph G can be derived from a voltage assignment in $C^1(G; n)$.

Characteristic polynomials of some graph coverings have already been computed. Chae, Kwak and Lee [**1**] have done it for double coverings of a graph. The characteristic polynomial of a graph covering when its voltages lie in an Abelian group or in a dihedral group was computed by Kwak and others [**3, 4**]. Mizuno and Sato [**5**] gave a formula for the characteristic polynomial of a regular covering. In this note, a formula for the characteristic polynomial of any (regular or irregular) graph covering is described, as an extension of all of the previous works.

Received 5th August, 2003
The first author is supported by NSF of China (No. 10001005) and the last two are supported by Com^{2}MaC-KOSEF in Korea.

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Let $\vec{G}$ denote the digraph obtained from G by replacing each edge of G with a pair of oppositely directed edges and let $\phi \in C^1(G,n)$. For each $\gamma \in S_n$, let $\vec{G}_{(\phi,\gamma)}$ denote the spanning subgraph of the digraph $\vec{G}$ whose directed edge set is $\phi^{-1}(\gamma)$. Let $V(G) = \{v_1, v_2, \ldots, v_m\}$ again. We define an order relation $\leqslant$ on $V(G^\phi)$ as follows: for $(v_i, s), (v_j, t) \in V(G^\phi)$, $(v_i, s) \leqslant (v_j, t)$ if and only if either $s < t$ or $s = t$ and $i \leqslant j$. Let $P(\gamma)$ denote the $n \times n$ permutation matrix associated with $\gamma \in S_n$, *that is*, its (s,t)-entry $P(\gamma)_{st} = 1$ if $s^\gamma = t$ and $P(\gamma)_{st} = 0$ otherwise. The tensor product $A \otimes B$ of the matrices A and B is considered as the matrix B having the element b_{st} replaced by the matrix Ab_{st}. Kwak and Lee ([**3**]) expressed the adjacency matrix $A(G^\phi)$ of a graph covering G^ϕ as

$$A(G^\phi) = \sum_{\gamma \in S_n} A(\vec{G}_{(\phi,\gamma)}) \otimes P(\gamma). \tag{1}$$

Let Γ be a finite group. A *representation* ρ of a group Γ over the complex field $\mathbb{C}$ is a group homomorphism from Γ to the general linear group $\mathrm{GL}(r, \mathbb{C})$ of invertible $r \times r$ matrices over $\mathbb{C}$. The number r is called the *degree* of the representation ρ (see [**6**]). Suppose that $\Gamma \leqslant S_n$ is a permutation group on Ω. It is clear that $P : \Gamma \to \mathrm{GL}(r, \mathbb{C})$ defined by $\gamma \to P(\gamma)$, where $P(\gamma)$ is the permutation matrix associated with $\gamma \in \Gamma$ corresponding to the action of Γ on Ω, is a representation of Γ. It is called the *permutation representation*. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ and let f_i be the degree of ρ_i for each $1 \leqslant i \leqslant \ell$, where $f_1 = 1$ and $\sum_{i=1}^{\ell} f_i^2 = |\Gamma|$. It is well-known [**6**] that the permutation representation P can be decomposed as the direct sum of irreducible representations. In other words, there exists an invertible matrix M such that

$$M^{-1}P(\gamma)M = \bigoplus_{i=1}^{\ell} (\rho_i(\gamma) \otimes I_{m_i}) \tag{2}$$

for any $\gamma \in \Gamma$, where $m_i \geqslant 0$ is the multiplicity of the irreducible representation ρ_i in the permutation representation P and $\sum_{i=1}^{\ell} m_i f_i = n$. Notice that m_1 is the number of orbits under the action of the group Γ on Ω. So $m_1 \geqslant 1$.

Now let $\phi \in C^1(G,n)$ and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle$, the subgroup generated by the voltages $\phi(e)$. Noting that $\sum_{i=1}^{\ell} m_i f_i = n$, from equations (1) and (2) we have

$$(I_m \otimes M)^{-1}(\lambda I_{mn} - A(G^\phi))(I_m \otimes M) = \bigoplus_{i=1}^{\ell} \Big[\Big(\lambda I_{mf_i} - \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \Big) \otimes I_{m_i} \Big].$$

Since $\rho_1(\gamma) = 1$ for any $\gamma \in \Gamma$ and $A(G) = \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)})$, we get

$$\Phi(G^\phi; \lambda) = \Phi(G; \lambda)^{m_1} \prod_{i=2}^{\ell} \Big[\det \Big(\lambda I_{mf_i} - \sum_{\gamma \in \Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \Big) \Big]^{m_i}.$$

Summarising our discussions, we have the following theorem.

Main Theorem. *Let G be a graph with m vertices, $\phi \in C^1(G,n)$ a voltage assignment on G and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle$. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of Γ and let f_i be the degree of ρ_i for each $1 \leqslant i \leqslant \ell$ with $f_1 = 1$. Then the characteristic polynomial of the n-fold covering G^ϕ of G derived from the voltage assignment ϕ is*

$$\Phi(G^\phi;\lambda) = \Phi(G;\lambda)^{m_1} \prod_{i=2}^{\ell} \left[\det\Big(\lambda I_{mf_i} - \sum_{\gamma\in\Gamma} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_i(\gamma) \Big) \right]^{m_i},$$

where m_i is the multiplicity of ρ_i in the permutation representation P of Γ.

Since $m_1 \geqslant 1$, it gives that for every covering graph G^ϕ of the graph G, the characteristic polynomial $\Phi(G;\lambda)$ is a divisor of the characteristic polynomial $\Phi(G^\phi;\lambda)$, in [**1**, Corollary 1]. When Γ is a regular subgroup of S_n, the permutation representation P of Γ is equivalent to the (right) regular representation and the covering G^ϕ is a regular covering of G. In this case, each multiplicity m_i is equal to f_i, the degree of the irreducible representation ρ_i. Therefore, Mizuno and Sato's [**5**, Theorem 2] can be derived from the main theorem. Furthermore, When Γ is Abelian or Γ is the dihedral group of order $2n$, the same results as in [**3**] and in [**4**] can also be deduced.

We close this note by giving a computational example which could not be done by using any formula that was known before. Let G be any graph, $\phi \in C^1(G,4)$ a voltage assignment on G and $\Gamma = \langle \phi(e) \mid e \in D(G) \rangle = S_4$. Note that the symmetric group S_4 can be generated by (12) and (1234). Then, the permutation representation P of S_4 can be decomposed by $P = \rho_1 \oplus \rho_2$, where $\rho_1 = 1$, the trivial representation, and ρ_2 is defined on the generators of Γ by

$$\rho_2\big((12)\big) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_2\big((1234)\big) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Therefore, the characteristic polynomial of the 4-fold covering G^ϕ of G derived from the voltage assignment ϕ is

$$\Phi(G^\phi;\lambda) = \Phi(G;\lambda) \det\Big(\lambda I_{3|V(G)|} - \sum_{\gamma\in S_4} A(\vec{G}_{(\phi,\gamma)}) \otimes \rho_2(\gamma) \Big). \tag{3}$$

For example, for the diamond graph G which is the complete graph K_4 minus an edge, one can see that

$$\Phi(G,\lambda) = \lambda(\lambda+1)(\lambda^2 - \lambda - 4).$$

Consider a voltage assignment ϕ which is defined as in Figure 1.

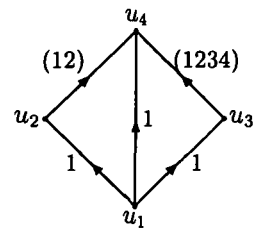

Figure 1: An S_4-voltage assignment ϕ on the diamond graph

From equation (3), one can get the characteristic polynomial of the graph G^ϕ as

$$\Phi(G^\phi;\lambda) = \Phi(G;\lambda)\lambda^2(\lambda^{10} - 12\lambda^8 + 2\lambda^7 + 51\lambda^6 - 22\lambda^5 - 87\lambda^4 + 66\lambda^3 + 39\lambda^2 - 54\lambda + 12).$$

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